Question 1: mean free path (10 points)

1. What is the mean free path?
2. If the mean free path is smaller than the size of the system, what are the important physical characteristics of the system?
3. Give an example of the system in the universe that the mean free path is larger than the size of the system.

Question 2: stellar configuration (20 points)

1. Hydrostatic equilibrium
   (a) Consider a small cylinder in the star, and give the equation of motion for this cylinder.
   (b) Assume the hydrostatic equilibrium, and derive the condition.
2. Mass profile of a star
   (a) A star has a density profile $\rho(r) = \rho_c \left(1 - \frac{r}{R}\right)$. Here, ρ_c is the core density, R is the radius of the star. Derive the mass profile $M(r)$ of this star. Assume the uniform composition.
   (b) Estimate the core density as a function of the total mass of the star M and the radius of the star R.
3. Core pressure of a star
   (a) Using the hydrostatic equilibrium equation, the density profile, and the mass profile of the star, derive the core pressure of the star P_c as a function of the total mass of the star M and the radius of the star R.
   (b) Calculate the core pressure for $M = 1M_\odot$ and $R = 1R_\odot$.
4. Core temperature of a star
   (a) Write down the equation of state for the ideal gas. Use the pressure P, density ρ, and temperature T.
   (b) Derive the core temperature of the star.
   (c) Calculate the core temperature for $M = 1M_\odot$ and $R = 1R_\odot$. Use 0.85 for the mean molecular weight of the Sun.

Question 3: The timescale τ for the physical quantity ϕ is defined as $\tau = \frac{\phi}{\dot{\phi}}$, where $\dot{\phi} = \frac{d\phi}{dt}$. (20 points)

1. Dynamical timescale

   - The escape velocity (or the free-fall velocity) is estimated from the balance between the kinetic energy and the gravitational potential energy. Show the escape velocity of the star with the mass M, radius R, and luminosity L.
   - Estimate the dynamical timescale $\tau_{dynamical}$ of the star with the mass M and the radius R. Normalize the dynamical timescale by the Solar quantities.
   - Compare the dynamical timescale of the Sun to the lifetime of the Sun. What are the interpretations of this result?

2. Thermal timescale

   - Estimate the thermal timescale $\tau_{thermal}$ of a star with the mass M, radius R, and luminosity L.
   - Calculate the thermal timescale of the Sun.
   - Compare the estimated thermal timescale of the Sun with the lifetime of the Sun. What are the interpretations of this result?

3. Nuclear timescale

   - Estimate the nuclear timescale $\tau_{nuclear}$ of a star with the mass M, radius R, and luminosity L.
   - Calculate the nuclear timescale of the Sun.
   - Compare the estimated nuclear timescale with the lifetime of the star. What are the interpretations of this result?

Question 4. Starting from radiative transfer equation, suppose the source function is a linear function of vertical optical depth $S_\lambda\left(\tau_{\lambda,\mathrm{v}}\right)=a_\lambda+b_\lambda\tau_{\lambda,\mathrm{v}}$

(a) Please explain what is limb darkening effect using the surface intensities $I_\lambda(\theta)$

(b) With additional assumptions of grey atmosphere, Eddington Approximation and local thermodynamic equilibrium, please derive the ratio of the emergent intensity at angle θ, $I(\theta)$ to that at the center of the star $I(\theta=0)$

(c) Explain how to use transit effect of an exoplanets to probe the limb darkening to its host star.

Question 5 (a) Write down the equations that describe the internal structure of stars. State clearly the meaning of each symbol, and the governing physical principle of each equation.

(b) Show that the outward pressure required to support a normal star of mass M and radius R is approximately GM^2/R^4.

(c) Considering a hypothetical one solar-mass star made of pure helium (no hydrogen or metals), compare as quantitatively as possible its physical parameters (e.g., luminosity, temperature structure, etc) with those of the Sun.

Question 6 In the conventional estimate for the Jeans mass, one may balance the gravitational pressure with the thermal pressure. However, star formation is now believed to be controlled by the presence of magnetic fields.

(a) Derive Jeans mass with presence of magnetic field with strength B

(b) Form (a) you may easily prove that the Jeans mass can become larger than the conventional estimate if the cloud has a magnetic field. Explain qualitatively about his fact.

(c) In strong field regime, show that the Jeans mass is only proportional to B^3/ρ^2

Question 7. A spherical static cloud with uniform initial density ρ_0 starts to collapse. Suppose this cloud is in absent of rotation, turbulence and the temperature of the gas is remains during the collapse. If the internal pressure gradient is too small to influence the motion appreciably, the equation of motion can be written as

$$\frac{d^2r}{dt^2}=-\frac{GM_r}{r^2}$$

where r is the distance from the center of the cloud and M_r is the mass of the sphere `interior to the radius r.

(a) If we are interested only for the surface that enclose M_r remains constant during the collapse, with proper initial conditions, show that the radius change rate of the surface can be written as

$$\frac{dr}{dt}=-\left[\frac{8\pi}{3}G\rho_0 r_0\left(\frac{r_0}{r}-1\right)\right]^{1/2}$$

where r_0 is the radius of the surface at $t=0$.

(b) Try to solve the equation about and prove the free-fall time scale can be written as

$$t_{ff}=\left(\frac{3\pi}{32G\rho_0}\right)^{1/2}$$